Solving For Internal Rate Of Return
Part III - The Newton-Raphson Method

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In Parts I and II we were given forward rates for an interst rate swap (variable rate leg) and were tasked with calculating the fixed rate equivalent (fixed rate leg) via the Bisection and Secant method, respectively. In Part III we will use the Newton-Raphson Method to calculate the fixed rate. To that end we will work through the following hypothetical problem from Parts I and II...

Our Hypothetical Problem

We are given the following interest rate swap parameters given that the swap has a notional value of \$1,000,000 and a term of 10 years with swap payments made at the end of each annual period.

Table 1: Interest Rate Swap Parameters

Annual Period	Forward Rate	Discount Factor	Variable Leg Payment	PV Variable Leg Payment	IRR 4.5338%
0	–	1.0000	–	–	-1,000,000.00
1	3.50%	0.9662	35,000.00	33,816.43	35,000.00
2	3.75%	0.9313	37,500.00	34,922.30	37,500.00
3	4.00%	0.8954	40,000.00	35,817.74	40,000.00
4	4.25%	0.8589	42,500.00	36,504.89	42,500.00
5	4.50%	0.8220	45,000.00	36,987.79	45,000.00
6	4.75%	0.7847	47,500.00	37,272.23	47,500.00
7	5.00%	0.7473	50,000.00	37,365.65	50,000.00
8	5.25%	0.7100	52,500.00	37,276.89	52,500.00
9	5.50%	0.6730	55,000.00	37,016.10	55,000.00
10	5.75%	0.6364	1,057,500.00	673,019.98	1,057,500.00
Total	–	–	1,462,500.00	1,000,000.00	462,500.00

Note: The IRR in the table above was derived by using the Excel IRR function.

Answer the following questions...

Question 1: Use the Newton-Raphson method to calculate the internal rate of return (i.e. the fixed rate).

Building Our Model

We will define the function $F(r)$ to be the net present value of a time series of cash flows discounted at the annual fixed interest rate r, the variable N to be the notional value of the swap, the variable C_t to be cash flow at the end of time t (variable leg payment), and the variable T to be swap term in years. The equation for the net present value of the swap at time zero is...

$$F(r) = \left[\sum_{t=1}^{T} C_t \left(1+r\right)^{-t}\right] - N \tag{1}$$

The Newton-Raphson method: Like the Bisection and Secant methods, the Newon-Raphson method is a root-finding method that applies to any continuous function that is monotonically increasing or decreasing. The method starts with a guess rate and then revises that rate after each iteration until function values convergence.

We will define the following discount rate variable definitions...

$$
\begin{aligned}
r &= \text{Actual discount rate where the NPV of the variable leg swap cash flows is zero.}\\
\hat{r} &= \text{Guess discount rate where the NPV of the variable leg swap cash flows is non-zero.}
\end{aligned} \tag{2}
$$

Using Equations (1) and (2) above, we will define the following functions...

$$
G(r) = N \text{ ...and... } G(\hat{r}) = \sum_{t=1}^{T} C_t \left(1 + \hat{r}\right)^{-t} \text{ ...and... } G'(\hat{r}) = \frac{\delta G(\hat{r})}{\delta \hat{r}} = \sum_{t=1}^{T} -t\, C_t \left(1 + \hat{r}\right)^{-(t+1)} \tag{3}
$$

The Newton-Raphson equation that we will iterate is... [1]

$$
r + \hat{\epsilon} = \hat{r} + \frac{G(r) - G(\hat{r})}{G'(\hat{r})} \tag{4}
$$

Given that the value of $G(r)$ is known, to solve for the variable r in that function we will iterate Equation (4) above as follows... [1]

Step	Task Description
1	Define the guess variable $\hat{r}$ such that $G(\hat{r})$ is close to $G(r)$ (the closer it is the less iterations).
2	Plug guess variable $\hat{r}$ into Equation (4) above and recalculate $r + \hat{\epsilon}$.
3	Using the results from Step 2 above redefine the guess variable to be $\hat{r} = r + \hat{\epsilon}$.
4	Go to Step 2 above and repeat.

The Answers To Our Hypothetical Problem

Question 1: Use the Newton-Raphson method to calculate the internal rate of return (i.e. the fixed rate).

We need to determine the start discount rate for our iteration process. To demonstrate the process we will define the start rate to be the minimum of the forward rates in Table 1 above. The start rate will be...

$$
\hat{r} = 3.50\% \tag{5}
$$

Using the start rate in Equation (5) above, the iteration process to 4 iterations is...

Iteration	r	r hat	G(r)	G(r hat)	dG(r hat)
1	4.4825%	3.5000%	1,000,000.00	1,087,672.66	-8,923,248.94
2	4.5337%	4.4825%	1,000,000.00	1,004,144.46	-8,094,735.68
3	4.5338%	4.5337%	1,000,000.00	1,000,010.44	-8,053,995.16
4	4.5338%	4.5338%	1,000,000.00	1,000,000.00	-8,053,892.31

Answer: At iteration 4 the discount rate converges to approximately 4.5338%.

References

[1] Gary Schurman, *The Newton-Raphson Method For Solving Non-Linear Equations,* October, 2009.